Exact solution of a class of critical dynamical systems: Information routing in complete graphs

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It is a controversial question to which extend scaling is observed in critical dynamical systems. Here we examine a class of dynamical systems describing the propagation of conserved quantities, such as the routing of information packages, and which are designed as critical systems. The long term information flow is governed by properties of the cyclic attractors in phase space.

We consider two classes of information routing models, Markovian routing without memory and vertex routing involving an one-step routing memory. Investigating the respective cycle length distributions for complete graphs we present analytic approximations which become exact in the limit of large numbers of vertices, the thermodynamic limit. We find log corrections to power-law scaling for the mean cycle length, as a function of the number of vertices, and a subpolynomial growth for the overall number of cycles.

Introduction.– The propagation of perturbations is a central notion in dynamical system theory. One speaks of a frozen state when a perturbation tends to die out, on the average, during the course of time evolution and of a chaotic state when perturbations tend to spread out [1, 2]. A given class of dynamical systems may change from frozen to chaotic behavior as a function of parameters, being critical right at the transition point.

At criticality, information is on the average conserved, as one can regard a perturbation of a state as the information about the persistence of small differences. A well studied example of a critical dynamical system is the Kauffman net with connectivity $K = 2$, an example of a random boolean network [3, 4]. In statistical mechanics critical systems are generically scale invariant [5], and it has been widely assumed that this statement would also hold for critical dynamical systems. Indeed numerical simulations seemed to support scaling in critical boolean networks, notably a $\sqrt{N}$ scaling for the number of attractors as a function of the number of vertices N had been proposed [3, 4].

An important clarification then came with the exact proof that the number of attractors actually grows faster than any power of N, and that the results of the numerical simulations suffered from systematic undersampling of phase space [6]. It could be shown, on the other side, that the number of frozen and the number of relevant nodes in a large class of critical boolean networks obeys powerlaw scaling [7].

The situation is then that certain properties of critical dynamical systems, at least for the case of random boolean networks, obey powerlaw scaling while others do not. It is hence important to investigate the possible occurrence of scaling in different classes of dynamical systems. We study here a novel class of dynamical systems modelling the transport of conserved quantities in network structures and which are hence critical per construction.

Transport on networks, like the spreading of rumors [8] and diseases [9] in social networks or the flow of capital in financial networks [10] is a basic process in biology [11], as well as in sociology and technical applications.

We considered here the transport of a conserved quantity, like information packages, in terms of routing processes. Information packages are sent from node to node and are routed at every vertex, as illustrated in Fig. 1. All routing process eventually ends up in one of the cyclic attractors. It has been shown previously that the geometric arrangement of the attractors on the network gives rise in the thermodynamic limit to a non-trivial distribution for the information centrality, which measures the number of attractors intersecting at a given vertex [12]. We present here analytic solutions for two types of vertex routing models, Markovian routing in the absence of a routing memory and vertex routing in the presence of an one-step memory.

The analytic solutions are asymptotically exact in the thermodynamic limit $N \to \infty$, they can be evaluated for large networks containing thousands to millions of sites. We present results for the scaling behavior of the overall number of attractors and for the mean of the cycle length distribution.

Models.– The two classes of models we consider differ with respect to the absence/presence of a routing memory. The phase space volume Ω is respectively linear and quadratic in the number of vertices N.

- For the Markovian model the selection of the next active vertex is independent of the previous state [13]. At every point in time only one vertex is active, the vertex with the information package. The phase space is hence identical with the collection of vertices; $\Omega = N$;

- For the vertex routing model the phase space is given by the collection of directed links; $\Omega = N(N-1)$. At every point in time one directed link is active, the link currently transporting the information package, compare Fig. 1.

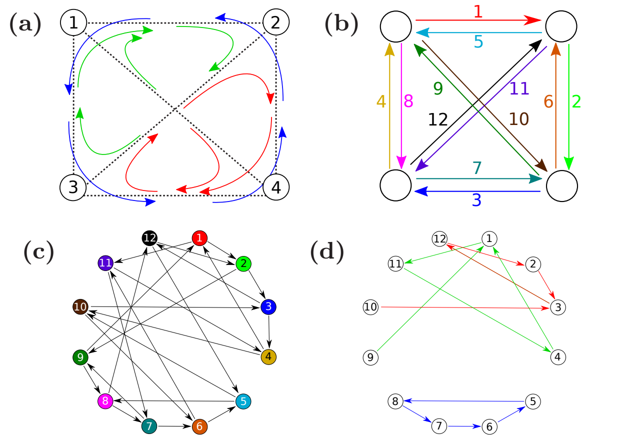

FIG. 1: Vertex routing dynamics for a $N=4$ complete graph **(a)** A realization of the routing tables. Routing through the first vertex follows $T_{312}=T_{213}=T_{214}=1$, with all other T_{i1j} vanishing. There are three cyclic attractors, namely (123), (243) and (1342). **(b)** Enumeration of all $N(N-1)=12$ directed edges, the phase-space elements. **(c)** The phase-space graph. **(d)** The same realization of the routing table as in **(a)**, now in terms of the phase-space graph.

In both setups the routing of information packages is realized through static routing tables. For every incoming edge the routing table specifies an allowed outgoing edge. A vertex k will transmit an information package, which was received from a vertex j, to a specific neighboring vertex i. The vertex routing table $\hat{T}$ corresponds to a tensor of binary elements $T_{ikj}=(\hat{T})_{ikj}\in\{0,1\}$,

$$T_{ikj} \;=\; \begin{cases} 0 & \text{no routing allowed} \\ 1 & \text{routing from} \qquad\quad \vec{e}_{jk} \text{ to } \vec{e}_{ki} \end{cases} , \qquad (1)$$

where $\vec{e}_{jk}$ denotes a directed edge from vertex j to vertex k. An example of a routing table for a four-site network is presented in Fig. 1.

We consider here critical models, viz models where the number of information packages is conserved. When the information is received along edge $\vec{e}_{jk}$, it can hence be transmitted along only one outgoing edge $\vec{e}_{ki}$,

$$\sum_i T_{ikj}=1, \qquad \sum_{ij} T_{ikj}=z_k \;, \qquad (2)$$

which is drawn randomly. Here z_k is the degree of vertex k, which is $N-1$ for fully connected networks considered here. For the Markovian model the routing table T_{ikj} is independent of j.

The two different routing models represent two limiting cases for the modelling of rumors in social networks. Markovian dynamics describes the spreading of rumors when the spreading is independent of social relations between the sender and the receiver. Vertex routing processes are, on the other side, important when the origin of the rumor is taken into consideration. A person receiving a work-related rumor might be prone to pass the rumor to another colleague, rather than to a family member or a friend.

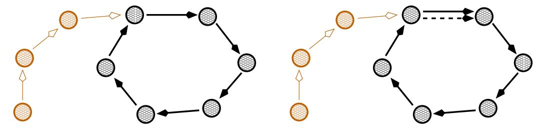

FIG. 2: Random walks through configuration space for the Markovian model (left) and for the vertex routing model (right). In order to find an attractor independent of the size of their basins of attraction (light color) one needs to close the path at the respective starting points.

Solution of the vertex routing model.– The dynamics consists of random walks through configuration space, as illustrated in Fig. 2. One can hence adapt the considerations [2], used for solving the Kauffman network for large connectivities $K\to\infty$, in order to solve the vertex routing model analytically.

At time-step $t=0,\,1,\,\ldots$ the number of phase-space elements visited previously is t. The probability that the next site is a site previously visited is then $t/(N-1)$. For the trajectory to close the routing has to retrace the existing path, which happens with probability $1/(N-1)$, compare Fig. 2.

The relative probability ρ_t of closing the path at time t is hence

$$\rho_t \;=\; \frac{t}{N-1}\,\frac{1}{N-1} \;. \qquad (3)$$

We define now with q_t the probability that the path is not closed at timestep t. The probability of having an open path at $t+1$ is then

$$q_{t+1} \;=\; q_t(1-\rho_t) \;=\; q_1\prod_{i=1}^{t}\frac{(N-1)^2-i}{(N-1)^2} \;,$$

with $q_1=1$, or

$$q_{t+1} \;=\; \frac{1}{(N-1)^{2t}}\,\frac{((N-1)^2-1)!}{((N-1)^2-t-1)!} \;. \qquad (4)$$

The average number $\langle C_L\rangle_v(N)$ of cycles of length $L\in[2,(N-1)^2+1]$ is then given by

$$\begin{aligned} \langle C_L\rangle_v(N) \;&=\; \frac{1}{L}\,\frac{N(N-1)}{(N-1)^2}\,q_{t=L-1} \qquad (5)\\ &=\; \frac{1}{L}\,\frac{N(N-1)}{(N-1)^{2L-2}}\,\frac{((N-1)^2-1)!}{((N-1)^2-L+1)!} \;. \end{aligned}$$

The factor $1/L$ takes care of overcounting, the factor $N(N-1)$ is the number of possible starting phase space elements, the factor $1/(N-1)^2$ the probability to close the path exactly at the starting phase space element and $q_{t=L-1}$ is the probability that a path containing $L=t+1$ directed links is open. Note that the path has to close at the starting phase space element and not at any other

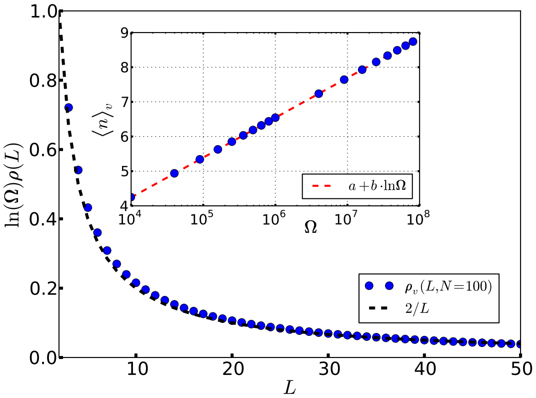

FIG. 3: The cycle length distributions $\rho_v(L)$, rescaled by $\log(\Omega)$, for the vertex routing model. The dashed line, $2/L$, represents the large-N and small-L limiting behavior. Inset: Log-Linear plot of the respective numbers of cycles, as a function of the phase space volume Ω.

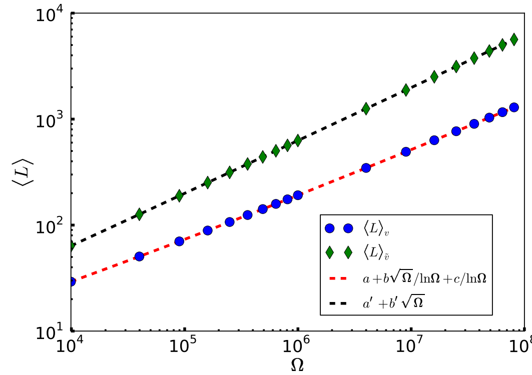

FIG. 4: The mean cycle lengths $\langle L \rangle_{v,\tilde{v}}$ for the vertex routing with quenched dynamics (blue circles) and the vertex routing with on the fly dynamics (green diamonds) respectively, as a function of the phase space volume Ω; log-log plot. The respective dashed lines are fits using $a+b\sqrt{\Omega}/\log(\Omega)+c/\log(\Omega)$ and $a' + b'\sqrt{\Omega}$, with a, a', b, b', c being fit parameters.

previously visited directed link. The probability to find an attractor would otherwise depend on the size of its basin of attraction, compare Fig. 2. The relation (5) is an approximation of the real cycle length distribution as it doesn't take into account corrections for self intersecting paths. These corrections drop however as $1/N$ and can be neglected in the thermodynamic limit. We denote with

$$\rho_v(L, N), \qquad \sum_L \rho_v(L, N) = 1$$

the normalized cycle length distribution for vertex routing with memory, which is obtained from (5) by normalizing $\langle C_L \rangle_v(N)$.

The cycle-length distribution for the Markovian model $\rho_m(L, N)$, can be derived in an analogous fashion, the phase space is here the collection of vertices. [12]

Intermodel scaling relation.– The number of cycles of length $L \in [2, N]$ for the Markovian model is [12]

$$\langle C_L \rangle_m(N) \;=\; \frac{1}{L}\,\frac{1}{(N-1)^L}\,\frac{N!}{(N-L)!}\,. \qquad (6)$$

TABLE I: Scaling with the number of vertices N, for the number of cycles and for the mean of the cycle length distribution, respectively for vertex routing (v) and the Markovian (m) model, and for the two types of dynamics, quenched and on the fly. Only relative quantities can be evaluated for on the fly dynamics.

		quenched	on the fly
(v)	number	$\log(N)$	–
	mean	$N/\log(N)$	N
(m)	number	$\log(N)$	–
	mean	$\sqrt{N}/\log(N)$	$\sqrt{N}$

Substituting N by $(N-1)^2+1$ in (6) we obtain for large N the scaling relation

$$\langle C_L \rangle_v(N) \;\sim\; \langle C_L \rangle_m((N-1)^2+1) \qquad (7)$$

between the number of cycles of the vertex routing and the Markovian model, $\langle C_L \rangle_v$ and $\langle C_L \rangle_m$.

Results.– Evaluating numerically the analytic expressions (5) and (6) for the number of cycles we find clear evidence for a logarithmic growth for the total number of attractors $\langle n \rangle_{v,m} = \sum_L \langle C_L \rangle_{v,m}$, as a function of phase space volume Ω, see Fig. 3. The total number of cycles hence grows slower than any polynomial of the number of vertices N, in contrast to critical Kauffman models, where it grows faster than any power of N [6].

The normalized cycle length distributions $\rho_{v,m}(L)$ thus scale as $1/\log(\Omega)$, due to the normalization. The rescaled distributions $\log(\Omega)\rho_{v,m}(L)$ approach the thermodynamic limit rapidly, compare Fig. 3. For small cycle lengths L the limiting functional form of the rescaled distributions is $2/L$, while for large $L \to L_{max}$ it falls off as $L_{max}^{(L_{max}-L+1/2)} \mathrm{e}^{-L_{max}}$. The limiting behavior of $\log(\Omega)\rho_{v,m}(L)$ is identical for both models, due to the intermodel scaling relation (7).

The determination of the scaling behavior is somewhat more subtle for the mean cycle length, see Fig. 4. We find that the functional dependence on the phase space volume is best reproduced by $a + b\sqrt{\Omega}/\log(\Omega) + c/\log(\Omega)$, where a, b, c are free parameters, which fits the data by about one order of magnitude better than a pure power law Ansatz $a' + b'\,\Omega^{c'}$. An overview of the obtained scaling relations is given in Table I. Note that in Figs. 3 and 4 we present only the data for the vertex routing model as it completely overlaps for large phase spaces Ω, due to the scaling (7), with the results for the Markovian model.

On the fly dynamics.– In addition to working with predetermined vertex routing tables one can generate dy-

namics 'on the fly' without explicitly creating routing tables for all vertices of the network.

For this kind of dynamics a routing for a given vertex is selected only when the trajectory visits this vertex. A cyclic attractor is then found when one state of the phase space (edge or node) is visited more then once. Then, one can show that the probability to find an L-cycle is $\tilde{\rho}_v(L,N) = \sum_{t=L}^{L_{max}} q_t$, where we denoted with $\tilde{\rho}_v(L,N)$ the weighted cycle length distributions for the vertex routing model and where q_t is given by (4). An analogous relation holds for the Markovian model. Generalizing the scaling relation (7) one finds $\tilde{\rho}_v(L,N) = \tilde{\rho}_m(L,(N-1)^2+1)$ and consequently $\langle L\rangle_{\tilde{v}}(N) = \langle L\rangle_{\tilde{m}}((N-1)^2+1)$, where $\tilde{\rho}_m$ denotes the weighted cycle length distributions for the Markovian model.

We find clear evidence for a scaling $\sim \sqrt{\Omega}$ for the mean cycle lengths, compare Fig. 4, and no indications for log corrections. Note that the overall number of cycles cannot be obtained when routing on the fly, only relative quantities can be evaluated.

Discussion and Conclusions.– For boolean networks the phase space volume Ω is 2^N and hence grows exponentially with the number of vertices N. The fact [6] that the number of attractors grows faster than any power of N could hence in principle be related to the exponential growth of the phase space volume. Whether these two scaling relations, the exponential growth of the phase space volume and the superpolynomial growth of the number of attractors, are indeed causally related is however not known to date.

Our results indicate that scaling in critical dynamical systems may generically be non-universal. For two solvable models with polynomial phase space scaling, as a function of the number of vertices, we find a subpolynomial growth in the number of attractors and log corrections to powerlaw scaling for the mean cycle length. These results indicate that there is probably no generic relation between the scaling of the phase space volume and the number of cycles.

We also note that other properties of critical dynamical systems, like the scaling of the number of frozen or relevant nodes for critical boolean networks [7], may show highly non-trivial behavior. For the case of vertex routing models one may define a measure of centrality, information centrality, determined by the number of attractors intersecting a given vertex, which scales to a non-trivial limiting distribution in the thermodynamic limit [12].

Our results may also be seen in the context of the surge in interests in modelling [14, 15] and in experimentally investigating [16, 17] the spontaneous neural dynamics of the brain. The observation of powerlaw scaling relations [18] have been interpreted as evidence of a critical self-organized neural state [19]. Our results indicate that there is no universal relation in dynamical systems theory between criticality and powerlaw scaling and that scaling is generically dependent on the observation modus. The unbiased statistics of a certain property, like the number of attractors or avalanches, may differ from a statistics obtained via stochastic sampling ($\rho_{v,m}(L)$ and $\tilde{\rho}_{v,m}(L)$ in our case). The later will in general be dependent on the size of the respective basins of attraction of the dynamical process considered, viz of a cycle or an avalanche. For the case of the vertex routing model studied here we found logarithmic corrections to powelaw scaling for the unbiased statistic and pure powerlaw scaling for stochastic on the fly sampling.

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